Neurčitý integrál $\int$

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↳ primitivní funkce

• Je dána funkce $f(x)$ definovaná v otevřeném intervalu I. Máme nalézt fci. $F(x)$ takovou, že pro každé $x \in I$ platí $F'(x) = f(x)$

Fce. $F(x)$ se nazývá primitivní funkce k $f(x)$ v I, nebo také neurčitý integrál fce. $f(x)$ v I.

$$\int f(x)\,dx = F(x)$$

$F(x) = \int f(x)\,dx \Rightarrow f(x) = F'(x)$; jelikož $(c)' = 0$ $\quad [F(x) + c]' = f_x$

c … integrační konstanta

- Jsou-li $F(x)$ a $G(x)$ dvě primitivní funkce k fci $f(x)$ v I, pak pro každé $x \in I$ platí $F(x) = G(x) + c$

• Ke každé fci. $f(x)$, která je v I spojitá existuje v ~~II~~ tomto intervalu funkce primitivní.

Vzorce:

$\int 0\,dx = c$ (= konstanta)

$\int dx = x + c$

$\int \sin x\,dx = -\cos x + c$

$\int e^x\,dx = e^x + c$

$\int \frac{1}{\cos^2 x}\,dx = \operatorname{tg} x + c$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad ; n \neq -1$

$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c$

$\int \cos x\,dx = \sin x + c$

$\int \frac{1}{\sin^2 x}\,dx = -\operatorname{cotg} x + c$

$\int k\,f(x)\,dx = k \int f(x)\,dx$

$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx + c$

$\int x^7 dx = \frac{x^8}{8} + c$ $\quad\quad$ $\int \frac{1}{x^4} dx = \int x^{-4} dx = \frac{x^{-3}}{-3} + c$

$\int \sqrt[3]{x^2} dx = \int x^{\frac{2}{3}} dx = \frac{x^{\frac{5}{3}}}{\frac{5}{3}}$

$\int 3x^2 \cdot \sqrt{x}\, dx = \cancel{\ldots} = 3\int x^{\frac{5}{2}} dx = 3 \frac{x^{\frac{7}{2}}}{\frac{7}{2}} = \frac{6}{7}\sqrt{x^7} = \frac{6}{7} x^3 \sqrt{x} + c$

$\int (x+2)^2 dx = \int (x^2+4x+4)\, dx = \int x^2 dx + \int 4x\, dx + \int 4\, dx = \frac{x^3}{3} + \frac{4x^2}{2} + 4x + c$

$= \frac{x^3}{3} + 2x^2 + 4x + c$

$\int \frac{3-x^2}{x^3} dx = \int (3x^{-3} - x^{-1})\, dx = 3\frac{x^{-2}}{-2} - \ln|x| = -\frac{3}{2} x^{-2} - \ln|x| + c$

$\int tg^2 x\, dx = \int \frac{\sin^2 x}{\cos^2 x} dx = \int \frac{1-\cos^2 x}{\cos^2 x} dx = \int \left(\frac{1}{\cos^2} - 1\right) dx = tg x - x + c$

$\int \frac{\sin 2x}{\cos x} dx = \int 2\sin x\, dx = -2\cos x + c$

$\int \frac{1}{\sin^2 x \cdot \cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x} = \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x} = \int \frac{1}{\cos^2 x} + \int \frac{1}{\sin^2 x} = tg x - cotg x +$

23.2.

DALŠÍ METODY VÝPOČTU

1) $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

$\int tg x\, dx = \int \frac{\sin x}{\cos x} = \cancel{\ldots} - \int \frac{-\sin x}{\cos x} = -\ln|\cos x| + c$

$\int \frac{x}{x^2-1} dx = \frac{1}{2}\int \frac{2x}{x^2-1} = \frac{1}{2}\ln|x^2-1| + c$

2) per partes (po částech)

$(u \cdot v)' = u' \cdot v + u v'$

$\int (u \cdot v)' = \int u'v + \int u v'$

$u \cdot v = \int u'v + \int u v'$

$\boxed{\int u'v = uv - \int u v'}$

$\int x \sin x\, dx = \left[\frac{x^2}{2}\sin x - \int \frac{x^2}{2}\cos x\, dx\right]$

$u' = x \Rightarrow u = \frac{x^2}{2}$

$v = \sin x \Rightarrow v' = \cos x$

$\int x \sin x\, dx = -\cos x \cdot x - \int (-\cos x \cdot 1)\, dx = -x\cos x + \sin x + c$

$u' = \sin x \Rightarrow u = -\cos x$

Př $\int(\sin^2 x)dx = \int(\sin x)(\sin x)\,dx = -\sin x\cos x - \int -\cos x\cos x = -\sin\frac{1}{2}x + \int\cos^2 x\,dx$

$u' = \sin x \Rightarrow u = -\cos x$

$v = \sin x \Rightarrow v' = +\cos x$

$= -\sin\frac{1}{2}x + \int(1-\sin^2 x)dx = -\sin\frac{1}{2}x + \int 1\,dx - \int\sin^2 x\,dx =$

$\sin x\cos x$

$= -\sin\frac{1}{2}x + x - \int\sin^2 x\,dx$

$2\int\sin^2 x\,dx = x - \sin x\cos x$

$\int\sin^2 x\,dx = \dfrac{x-\sin\cos x}{2} + c$

Př $\int\dfrac{-\sin x}{1+\cos x}dx = -\int\dfrac{-\sin x}{1+\cos x} = -\ln|1+\cos x| + c$

$(1+\cos x)' = -\sin x$

Př $\int(\ln x)dx = \int(1\ln x)dx = x\ln x - \int x\frac{1}{x} = \underline{x\ln x - x + c}$

$u' = 1 \Rightarrow u = x$

$v = \ln x \Rightarrow v' = \frac{1}{x}$

③ substituční metoda

Př $\int\sin(5x+3)\,dx = \int\frac{\sin(5x+3)}{5}\underbrace{(5\,dx)}_{dt} = \frac{1}{5}\int\sin t\,dt$

$t = 5x+3$

1.der. $\dfrac{dt}{dx} = 5 \qquad dt = 5dx$

$= \frac{1}{5}(-\cos t) = -\frac{1}{5}\cos(5x+3) + c$

POZN: $y' = \dfrac{dy}{dx}$ $\left(\dfrac{\Delta y}{\Delta x}\right)$

dy, dx ... diferenciály

Př $\int\sin 2x\,dx$

1/ $t = 2x$

$\frac{dt}{dx} = 2 \Rightarrow dt = 2dx$

$\frac{1}{2}\int\sin t\cdot 2dt = -\frac{1}{2}\cos 2x + c = -\frac{1}{2}(\cos^2 x - \sin^2 x) = -\frac{1}{2}(1-\sin^2 x-\sin^2 x) = -\frac{1}{2}+$ ...

ZK: $\left(-\frac{1}{2}\cos 2x\right)' = -\frac{1}{2}(-\sin 2x)2 = \sin 2x$ ✓

2/ $\int 2\sin x\cos x\,dx = 2\int\sin x\cos x\,dx = 2\int t\frac{\cos x}{\cos x}dt = 2\frac{t^2}{2} = \sin^2 x + c$

$t = \sin x$

$\frac{dt}{dx} = +\cos x \Rightarrow dt = dx(+\cos x)$

ZK: $(\sin^2 x)' = 2\sin x\cos x = \sin 2x$ ✓

3/ per partes $2\int\sin x\cos x = 2(-\cos^2 x - \int\sin x\cos x) = -2\cos^2 x - 2\int\sin x\cos x\,dx = -2\cos^2 x - \int 2\sin x\cos$ ...

$u' = \sin x \quad u = -\cos x$